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SHADOWS OF WAVE FRONTS AND ARNOLD-BENNEQUIN TYPE INVARIANTS OF FRONTS ON SURFACES AND ORBIFOLDS

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ABSTRACT. A first order Vassiliev invariant of an oriented knot in an S^1 -fibration and a Seifert fibration over a surface is constructed. It takes values in a quotient of the group ring of the first homology group of the total space of the fibration. It gives rise to an invariant of wave fronts on surfaces and orbifolds related to the Bennequin-type invariants of the Legendrian curves studied by F. Aicardi, V. Arnold, M. Polyak, and S. Tabachnikov. Formulas expressing these relations are presented.

We also calculate Turaev's shadow for the Legendrian lifting of a wave front. This allows to use in the case of wave fronts all invariants known for shadows.

Most of the proofs in this paper are postponed until the last section.

Everywhere in this text an S^1 -fibration is a locally-trivial fibration with fibers homeomorphic to S^1 .

In this paper the multiplicative notation for the operation in the first homology group is used. The zero homology class is denoted by e . The reason for this is that we have to deal with the integer group ring of the first homology group. For a group G the group of all formal half-integer linear combinations of elements of G is denoted by $\frac{1}{2}\mathbb{Z}[G]$.

We work in the differential category.

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1. INTRODUCTION

In [5] Polyak suggested a quantization $l_q(L) \in \frac{1}{2}\mathbb{Z}[q, q^{-1}]$ of the Bennequin invariant of a generic cooriented oriented wave front $L \subset \mathbb{R}^2$. In this paper we construct an invariant $S(L)$ which is, in a sense, a generalization of $l_q(L)$ to the case of a wave front on an arbitrary surface F .

In the same paper [5] Polyak introduced Arnold's [3] J^+ -type invariant of a front L on an oriented surface F . It takes values in $H_1(ST^*F, \frac{1}{2}\mathbb{Z})$. We show that $S(L) \in \frac{1}{2}\mathbb{Z}[H_1(ST^*F)]$ is a refinement of this invariant in the sense that it is taken to Polyak's invariant under the natural mapping $\frac{1}{2}\mathbb{Z}[H_1(ST^*F)] \rightarrow H_1(ST^*F, \frac{1}{2}\mathbb{Z})$.

Further we generalize $S(L)$ to the case where L is a wave front on an orbifold.

Invariant $S(L)$ is constructed in two steps. The first one consists in lifting of L to the Legendrian knot λ in the S^1 -fibration $\pi : ST^*F \rightarrow F$. The second step can be applied to any knot in an S^1 -fibration, and it involves the structure of the fibration in a crucial way. This step allows us to define the S_K invariant of a knot

K in the total space N of an S^1 -fibration. Since ordinary knots are considered up to a rougher equivalence relation (ordinary isotopy versus Legendrian isotopy), in order for S_K to be well defined it has to take values in a quotient of $\mathbb{Z}[H_1(N)]$. This invariant is generalized to the case of a knot in a Seifert fibration, and this allows us to define $S(L)$ for wave fronts on orbifolds.

All these invariants are Vassiliev invariants of order one in an appropriate sense.

For each of these invariants we introduce its version with values in the group of formal linear combinations of the free homotopy classes of oriented curves in the total space of the corresponding fibration.

The first invariants of this kind were constructed by Fiedler [4] in the case of a knot K in a \mathbb{R}^1 -fibration over a surface and by Aicardi in the case of a generic oriented cooriented wave front $L \subset \mathbb{R}^2$. The connection between these invariants and S_K is discussed in [9].

The space ST^*F is naturally fibered over a surface F with a fiber S^1 . In [10] Turaev introduced a shadow description of a knot K in an oriented three dimensional manifold N fibered over an oriented surface with a fiber S^1 . A shadow presentation of a knot K is a generic projection of K , together with an assignment of numbers to regions. It describes a knot type modulo a natural action of $H_1(F)$. It appeared to be a very useful tool. Many invariants of knots in S^1 -fibrations, in particular quantum state sums, can be expressed as state sums for their shadows. In this paper we construct shadows of Legendrian liftings of wave fronts. This allows one to use any invariant already known for shadows in the case of wave fronts.

However, in this paper shadows are used mainly for the purpose of depicting knots in S^1 -fibrations.

2. SHADOWS

2.1. Preliminary constructions. We say that a one-dimensional submanifold L of a total space N^3 of a fibration $\pi : N^3 \rightarrow F^2$ is *generic with respect to π* if $\pi|_L$ is a generic immersion. Recall that an immersion of 1-manifold into a surface is said to be *generic* if it has neither self-intersection points with multiplicity greater than 2 nor self-tangency points, and at each double point its branches are transversal to each other. An immersion of (a circle) S^1 to a surface is called a *curve*.

Let π be an oriented S^1 -fibration of N over an oriented closed surface F .

N admits a fixed point free involution which preserves fibers. Let \tilde{N} be the quotient of N by this involution, and let $p : N \rightarrow \tilde{N}$ be the corresponding double covering. Each fiber of p (a pair of antipodal points) is contained in a fiber of π . Therefore, π factorizes through p and we have a fibration $\tilde{\pi} : \tilde{N} \rightarrow F$. Fibers of $\tilde{\pi}$ are projective lines. They are homeomorphic to circles.

An isotopy of a link $L \subset N$ is said to be vertical with respect to π if each point of L moves along a fiber of π . It is clear that if two links are vertically isotopic, then their projections coincide. Using vertical isotopy we can modify each generic link L in such a way that any two points of L belonging to the same fiber lie in the same orbit of the involution. Denote the obtained generic link by L' .

Let $\tilde{L} = p(L')$. It is obtained from L' by gluing together points lying over the same point of F . Hence $\tilde{\pi}$ maps \tilde{L} bijectively to $\pi(L) = \pi(L')$. Let $r : \pi(L) \rightarrow \tilde{L}$ be an inverse bijection. It is a section of $\tilde{\pi}$ over $\pi(L)$.

For a generic non-empty collection of curves on a surface by a *region* we mean the closure of a connected component of the complement of this collection. Let X

be a region for $\pi(L)$ on F . Then $\tilde{\pi}|_X$ is a trivial fibration. Hence we can identify it with the projection $S^1 \times X \rightarrow X$. Let ϕ be a composition of the section $r|_{\partial X}$ with the projection to S^1 . It maps ∂X to S^1 . Denote by α_X the degree of ϕ . (This is actually an obstruction to an extension of $r|_{\partial X}$ to X .) One can see that α_X does not depend on the choice of the trivialization of $\tilde{\pi}$ and on the choice of L' .

2.2. Basic definitions and properties.

Definition 2.2.1. The number $\frac{1}{2}\alpha_X$ corresponding to a region X is called the *gleam* of X and is denoted by $\text{gl}(X)$. A *shadow* $s(L)$ of a generic link $L \subset N$ is a (generic) collection of curves $\pi(L) \subset F$ with the gleams assigned to each region X . The sum of gleams over all regions is said to be the *total gleam* of the shadow.

2.2.2. One can check that for any region X the integer α_X is congruent modulo 2 to the number of corners of X . Therefore, $\text{gl}(X)$ is an integer if the region X has even number of corners and half-integer otherwise.

2.2.3. The total gleam of the shadow is equal to the Euler number of π .

Definition 2.2.4. A *shadow* on F is a generic collection of curves together with the numbers $\text{gl}(X)$ assigned to each region X . These numbers can be either integers or half-integers, and they should satisfy the conditions of 2.2.2 and 2.2.3.

There are three local moves S_1, S_2 , and S_3 of shadows shown in Figure 1. They are similar to the well-known Reidemeister moves of planar knot diagrams.

Definition 2.2.5. Two shadows are said to be *shadow equivalent* if they can be transformed to each other by a finite sequence of moves S_1, S_2, S_3 , and their inverses.

2.2.6. There are two more important shadow moves \tilde{S}_1 and \tilde{S}_3 shown in Figure 2. They are similar to the previous versions of the first and the third Reidemeister moves and can be expressed in terms of S_1, S_2, S_3 , and their inverses.

2.2.7. In [10] the action of $H_1(F)$ on the set of all isotopy types of links in N is constructed as follows. Let L be a generic link in N and β an oriented (possibly self-intersecting) curve on F presenting a homology class $[\beta] \in H_1(F)$. Deforming β we can assume that β intersects $\pi(K)$ transversally at a finite number of points different from the self-intersection points of $\pi(K)$. Denote by $\alpha = [a, b]$ a small segment of L such that $\pi(\alpha)$ contains exactly one intersection point c of $\pi(L)$ and β . Assume that $\pi(a)$ lies to the left, and $\pi(b)$ to the right of β . Replace α by the arc α' shown in Figure 3. We will call this transformation of L a *fiber fusion* over the point c . After we apply fiber fusion to L over all points of $\pi(L) \cap \beta$ we get a new generic link L' with $\pi(L) = \pi(L')$. One can notice that the shadows of K and K' coincide. Indeed, each time β enters a region X of $s(L)$, it must leave it. Hence the contributions of the newly inserted arcs to the gleam of X cancel out. Thus links belonging to one $H_1(F)$ -orbit always produce the same shadow-link on F .

Theorem 2.2.8 (Turaev [10]). *Let N be an oriented closed manifold, F an oriented surface, and $\pi : N \rightarrow F$ an S^1 -fibration with the Euler number $\chi(\pi)$. The mapping that associates to each link $L \subset N$ its shadow equivalence class on F establishes a bijective correspondence between the set of isotopy types of links in N modulo the action of $H_1(F)$ and the set of all shadow equivalence classes on F with the total gleam $\chi(\pi)$.*

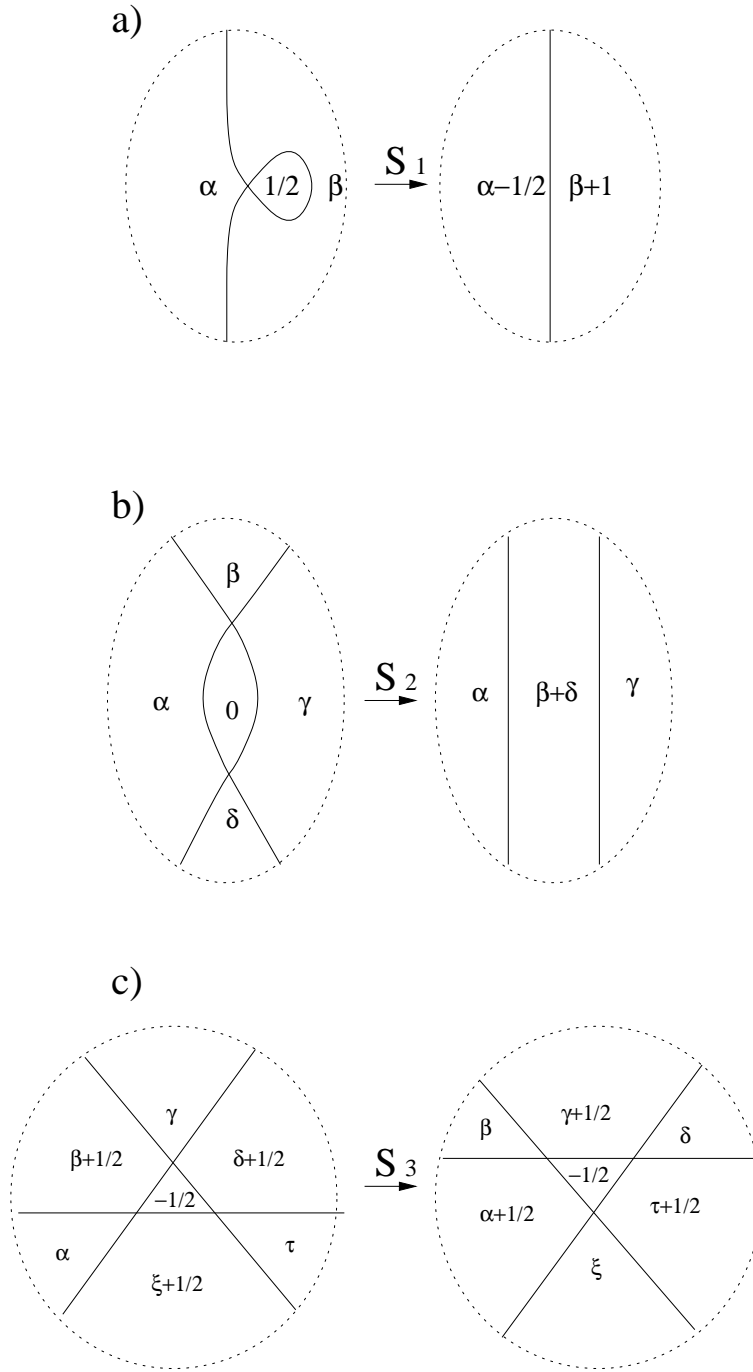


FIGURE 1.

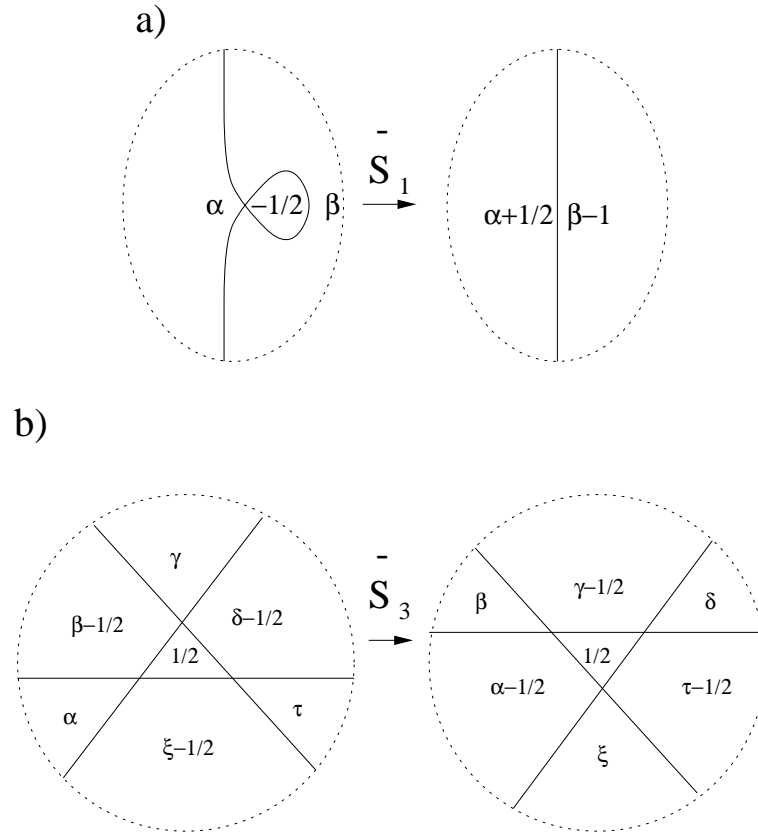


FIGURE 2.

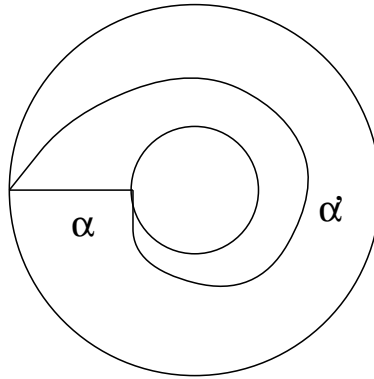


FIGURE 3.

2.2.9. It is easy to see that all links whose projections represent $0 \in H_1(F)$ and whose shadows coincide are homologous to each other. To prove this, one looks at the description of a fiber fusion and notices that to each fiber fusion where we add a

positive fiber corresponds another where we add a negative one. Thus the numbers of positively and negatively oriented fibers we add are equal, and they cancel out.

2.2.10. As it was remarked in [10] it is easy to transfer the construction of shadows and Theorem 2.2.8 to the case where F is a non-closed oriented surface and N is an oriented manifold. In order to define the gleams of the regions that have a non-compact closure or contain components of ∂F , we have to choose a section of the fibration over all boundary components and ends of F . In the case of non-closed F the total gleam of the shadow is equal to the obstruction to the extension of the section to the entire surface.

3. INVARIANTS OF KNOTS IN S^1 -FIBRATIONS.

3.1. Main constructions. In this section we deal with knots in an S^1 -fibration π of an oriented three-dimensional manifold N over an oriented surface F . We do not assume F and N to be closed. As it was said in 2.2.10, all theorems from the previous section are applicable in this case.

Definition 3.1.1 (of S_K). Orientations of N and F determine an orientation of a fiber of the fibration. Denote by $f \in H_1(N)$ the homology class of a positively oriented fiber.

Let $K \subset N$ be an oriented knot which is generic with respect to π . Let v be a double point of $\pi(K)$. The fiber $\pi^{-1}(v)$ divides K into two arcs that inherit the orientation from K . Complete each arc of K to an oriented knot by adding the arc of $\pi^{-1}(v)$ such that the orientations of these two arcs define an orientation of their union. The orientations of F and $\pi(K)$ allow one to identify a small neighborhood of v in F with a model picture shown in Figure 4a. Denote the knots obtained by the operation above by μ_v^+ and μ_v^- as shown in Figure 4. We will often call this construction a *splitting* of K (with respect to the orientation of K).

This splitting can be described in terms of shadows as follows. Note that μ_v^+ and μ_v^- are not in general position. We slightly deform them in a neighborhood of $\pi^{-1}(v)$, so that $\pi(\mu_v^+)$ and $\pi(\mu_v^-)$ do not have double points in the neighborhood of v . Let P be a neighborhood of v in F homeomorphic to a closed disk. Fix a section over ∂P such that the intersection points of $K \cap \pi^{-1}(\partial P)$ belong to the section. Inside P we can construct Turaev's shadow (see 2.2.10). The action of $H_1(\text{Int } P) = e$ on the set of the isotopy types of links is trivial. Thus the part of K can be reconstructed in the unique way (up to an isotopy fixed on ∂P) from the shadow over P (see 2.2.10). The shadows for μ_v^+ and μ_v^- are shown in Figures 4a and 4b respectively.

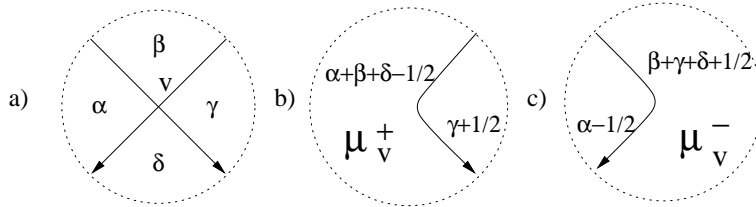


FIGURE 4.

Regions for the shadows $s(\mu_v^+)$ and $s(\mu_v^-)$ are, in fact, unions of regions for $s(K)$. One should think of gleams as of measure, so that the gleam of a region is the sum of all numbers inside.

Let H be the quotient of the group ring $\mathbb{Z}[H_1(N)]$ (viewed as a \mathbb{Z} -module) by the submodule generated by $\{[K] - f, [K]f - e\}$. Here by $[K] \in H_1(N)$ we denote the homology class represented by the image of K .

Finally define $S_K \in H$ by the following formula, where the summation is taken over all double points v of $\pi(K)$:

$$S_K = \sum_v ([\mu_v^+] - [\mu_v^-]). \quad (1)$$

3.1.2. Since $\mu_v^+ \cup \mu_v^- = K \cup \pi^{-1}(v)$ we have

$$[\mu_v^+][\mu_v^-] = [K]f. \quad (2)$$

Theorem 3.1.3. *S_K is an isotopy invariant of the knot K .*

For the proof of Theorem 3.1.3 see Subsection 7.1.

3.1.4. It follows from 3.1.2 that S_K can also be described as an element of $\mathbb{Z}[H_1(N)]$ equal to the sum of $([\mu_v^+] - [\mu_v^-])$ over all double points for which the sets $\{[\mu_v^+], [\mu_v^-]\}$ and $\{e, f\}$ are disjoint. Note that in this case we do not need to factorize $\mathbb{Z}[H_1(N)]$ to make S_K well defined.

3.1.5. One can obtain an invariant similar to S_K with values in the free \mathbb{Z} -module generated by the set of all free homotopy classes of oriented curves in N . To do this one substitutes the homology classes of μ_v^+ and μ_v^- in (1) with their free homotopy classes and takes the summation over the set of all double points v of $\pi(K)$ such that neither one of the knots μ_v^+ and μ_v^- is homotopic to a trivial loop and neither one of them is homotopic to a positively oriented fiber (see 3.1.4).

To prove that this is indeed an invariant of K one can easily modify the proof of Theorem 3.1.3.

3.2. S_K is a Vassiliev invariant of order one.

3.2.1. Let $\pi : N \rightarrow F$ be an S^1 -fibration over a surface. Let $K \subset N$ be a knot generic with respect to π and v a double point of $\pi(K)$. A modification of pushing of one branch of K through the other along the fiber $\pi^{-1}(v)$ is called a *modification of K along the fiber $\pi^{-1}(v)$* .

3.2.2. If a fiber fusion increases by one the gleam γ in Figure 4b, then $[\mu_v^+]$ is multiplied by f . If a fiber fusion increases by one the gleam α in Figure 4c, then $[\mu_v^-]$ is multiplied by f^{-1} . These facts are easy to verify.

3.2.3. Let us find out how S_K changes under the modification along a fiber over a double point v . Consider a singular knot K' (whose only singularity is a point v of transverse self-intersection). Let ξ_1 and ξ_2 be the homology classes of the two loops of K' adjacent to v . The two resolutions of this double point correspond to adding $\pm \frac{1}{2}$ to the gleams of the regions adjacent to v in two ways shown in Figures 5b and 5c.

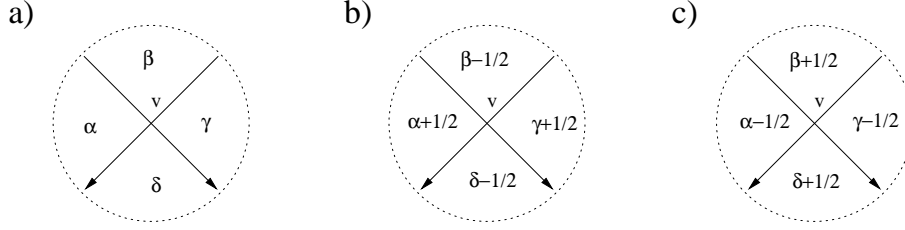


FIGURE 5.

Using 3.2.2 one verifies that under the corresponding modification S_K changes by

$$(f - e)(\xi_1 + \xi_2). \quad (3)$$

This means that the first derivative of S_K depends only on the homology classes of the two loops adjacent to the singular point. Hence the second derivative of S_K is 0. Thus it is a Vassiliev invariant of order one in the usual sense.

For similar reasons the version of S_K with values in the free \mathbb{Z} -module generated by all free homotopy classes of oriented curves in N is also a Vassiliev invariant of order one.

Theorem 3.2.4.

I: If K and K' are two knots representing the same free homotopy class, then S_K and $S_{K'}$ are congruent modulo the submodule generated by elements of form

$$(f - e)(j + [K]j^{-1}) \quad (4)$$

for $j \in H_1(N)$.

II: If K is a knot, and $S \in H$ is congruent to S_K modulo the submodule generated by elements of form (4) (for $j \in H_1(N)$), then there exists a knot K' such that:

- a: K and K' represent the same free homotopy class;
- b: $S_{K'} = S$.

For the proof of Theorem 3.2.4 see subsection 7.2.

3.3. Example. If N is a solid torus T fibered over a disk, then we can calculate the value of S_K directly from the shadow of K .

Definition 3.3.1. Let C be an oriented closed curve in \mathbb{R}^2 and X a region for C . Take a point $x \in \text{Int } X$ and connect it to a point near infinity by a generic oriented path D . Define the sign of an intersection point of C and D as shown in Figure 6. Let $\text{ind}_C X$ be the sum over all intersection points of C and D of the signs of these points.

It is easy to see that $\text{ind}_C(X)$ is independent on the choices of x and D .

Definition 3.3.2. Let $K \subset T$ be an oriented knot which is generic with respect to π , and let $s(K)$ be its shadow. Define $\sigma(s(K)) \in \mathbb{Z}$ as the following sum over all



FIGURE 6.

regions X for $\pi(K)$:

$$\sigma(s(K)) = \sum_X \text{ind}_{\pi(K)}(X) \text{gl}(X). \quad (5)$$

Denote by $h \in \mathbb{Z}$ the image of $[K]$ under the natural identification of $H_1(T)$ with \mathbb{Z} .

Lemma 3.3.3. $\sigma(s(K)) = h$.

3.3.4. Put

$$S'_K = \sum t^{\sigma(s(\mu_v^+))} - t^{\sigma(s(\mu_v^-))}, \quad (6)$$

where the sum is taken over all double points v of $\pi(K)$ such that $\{0, 1\}$ and $\{\sigma(s(\mu_v^+)), \sigma(s(\mu_v^-))\}$ are disjoint (see 3.1.4).

Lemma 3.3.3 implies that S'_K is the image of S_K under the natural identification of $\mathbb{Z}[H_1(T)]$ with the ring of finite Laurent polynomials (see 3.1.4).

One can show [9] that S'_K and Aicardi's partial linking polynomial of K (which was introduced in [1]) can be explicitly expressed in terms of each other.

3.4. Further generalizations of the S_K invariant. One can show that an invariant similar to S_K can be introduced in the case where N is oriented and F is non-orientable.

Definition 3.4.1 (of \tilde{S}_K). Let N be oriented and F non-orientable. Let $K \subset N$ be an oriented knot generic with respect to π , and let v be a double point of $\pi(K)$. Fix an orientation of a small neighborhood of v in F . Since N is oriented this induces an orientation of the fiber $\pi^{-1}(v)$. Similarly to the definition of S_K (see 3.1.1), we split our knot with respect to the orientation and obtain two knots $\mu_1^+(v)$ and $\mu_1^-(v)$. Then we take the other orientation of the neighborhood of v in F , and in the same way we obtain another pair of knots $\mu_2^+(v)$ and $\mu_2^-(v)$. The element $([\mu_1^+(v)] - [\mu_1^-(v)] + [\mu_2^+(v)] - [\mu_2^-(v)]) \in \mathbb{Z}[H_1(N)]$ does not depend on which orientation of the neighborhood of v we choose first.

Similarly to the definition of S_K , we can describe all this in terms of shadows as it is shown in Figure 7. These shadows are constructed with respect to the same orientation of the neighborhood of v .

Let f be the homology class of a fiber of π oriented in some way. As one can easily prove $f^2 = e$, so it does not matter which orientation we choose to define f . Let \tilde{H} be the quotient of $\mathbb{Z}[H_1(N)]$ (viewed as a \mathbb{Z} -module) by the \mathbb{Z} -submodule

generated by $\{[K] - f + e - [K]f = (e - f)([K] + e)\}$. Finally define $\tilde{S}_K \in \tilde{H}$ by the following formula, where the summation is taken over all double points v of $\pi(K)$:

$$\tilde{S}_K = \sum_v \left([\mu_1^+(v)] - [\mu_1^-(v)] + [\mu_2^+(v)] - [\mu_2^-(v)] \right). \quad (7)$$

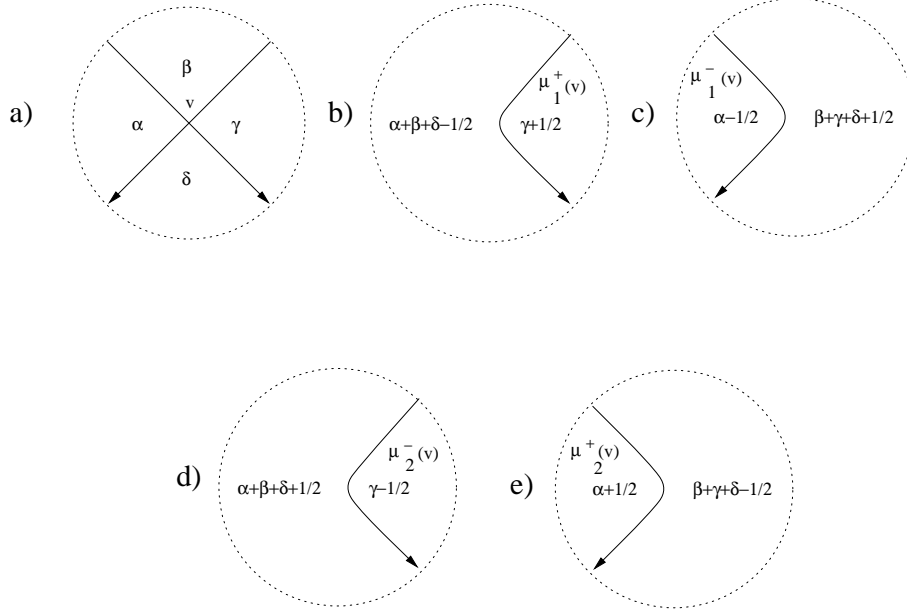


FIGURE 7.

Theorem 3.4.2. \tilde{S}_K is an isotopy invariant of the knot K .

The proof is essentially the same as the proof of Theorem 3.1.3.

3.4.3. One can easily prove that \tilde{S}_K invariant satisfies relations similar to (3). In particular, \tilde{S}_K is also a Vassiliev invariant of order one.

One can introduce a version of this invariant with values in the free \mathbb{Z} -module generated by all free homotopy classes of oriented curves in N . To do this, we substitute the homology classes of $\mu_1^+(v)$, $\mu_1^-(v)$, $\mu_2^+(v)$, and $\mu_2^-(v)$ with the corresponding free homotopy classes. The summation should be taken over the set of all double points of $\pi(K)$ for which neither one of $\mu_1^+(v)$, $\mu_1^-(v)$, $\mu_2^+(v)$, and $\mu_2^-(v)$ is homotopic to a trivial loop and neither one of them is homotopic to a fiber of π . To prove that this is indeed an invariant of K , one easily modifies the proof of Theorem 3.1.3.

4. INVARIANTS OF KNOTS IN SEIFERT FIBERED SPACES

Let (μ, ν) be a pair of relatively prime integers. Let

$$D^2 = \left\{ (r, \theta); 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi \right\} \subset \mathbb{R}^2$$

be the unit disk defined in polar coordinates. A fibered solid torus of type (μ, ν) is the quotient space of the cylinder $D^2 \times I$ via the identification $((r, \theta), 1) = ((r, \theta + \frac{2\pi\nu}{\mu}), 0)$. The fibers are the images of the curves $x \times I$. The integer μ is called the index or the multiplicity. For $|\mu| > 1$ the fibered solid torus is said to be *exceptionally fibered*, and the fiber that is the image of $0 \times I$ is called the *exceptional fiber*. Otherwise the fibered solid torus is said to be *regularly fibered*, and each fiber is a *regular fiber*.

Definition 4.0.4. An orientable three manifold S is said to be a *Seifert fibered manifold* if it is a union of pairwise disjoint closed curves, called fibers, such that each one has a closed neighborhood which is a union of fibers and is homeomorphic to a fibered solid torus by a fiber preserving homeomorphism.

A fiber h is called *exceptional* if h has a neighborhood homeomorphic to an exceptionally fibered solid torus (by a fiber preserving homeomorphism), and h corresponds via the homeomorphism to the exceptional fiber of the solid torus. If $\partial S \neq \emptyset$, then ∂S should be a union of regular fibers.

The quotient space obtained from a Seifert fibered manifold S by identifying each fiber to a point is a 2-manifold. It is called the orbit space and the images of the exceptional fibers are called *the cone points*.

4.0.5. For an exceptional fiber a of an oriented Seifert fibered manifold there is a unique pair of relatively prime integers (μ_a, ν_a) such that $\mu_a > 0$, $|\nu_a| < \mu_a$, and a neighborhood of a is homeomorphic (by a fiber preserving homeomorphism) to a fibered solid torus of type (μ_a, ν_a) . We call the pair (μ_a, ν_a) *the type of the exceptional fiber a* . We also call this pair the type of the corresponding cone point.

We can define an invariant of an oriented knot in a Seifert fibered manifold that is similar to the S_K invariant.

Clearly any S^1 -fibration can be viewed as a Seifert fibration without cone points. This justifies the notation in the definition below.

Definition 4.0.6 (of S_K). Let N be an oriented Seifert fibered manifold with an oriented orbit space F . Let $\pi : N \rightarrow F$ be the corresponding fibration and $K \subset N$ an oriented knot in general position with respect to π . Assume also that K does not intersect the exceptional fibers. For each double point v of $\pi(K)$ we split K into μ_v^+ and μ_v^- (see 3.1.1). Let A be the set of all exceptional fibers. Since N and F are oriented, we have an induced orientation of each exceptional fiber $a \in A$. For $a \in A$ set f_a to be the homology class of the fiber with this orientation. For $a \in A$ of type (μ_a, ν_a) (see 4.0.5) set $N_1(a) = \{k \in \{1, \dots, \mu_a\} \mid \frac{2\pi k \nu_a}{\mu_a} \bmod 2\pi \in (0, \pi]\}$, $N_2(a) = \{k \in \{1, \dots, \mu_a\} \mid \frac{2\pi k \nu_a}{\mu_a} \bmod 2\pi \in (0, \pi)\}$. Define $R_a^1, R_a^2 \in \mathbb{Z}[H_1(N)]$ by the following formulas:

$$R_a^1 = \sum_{k \in N_1(a)} ([K] f_a^{\mu_a - k} - f_a^k) - \sum_{k \in N_2(a)} (f_a^{\mu_a - k} - [K] f_a^k), \quad (8)$$

$$R_a^2 = \sum_{k \in N_1(a)} (f_a^{\mu_a - k} - [K] f_a^k) - \sum_{k \in N_2(a)} ([K] f_a^{\mu_a - k} - f_a^k). \quad (9)$$

Let H be the quotient of $\mathbb{Z}[H_1(N)]$ (viewed as a \mathbb{Z} -module) by the free \mathbb{Z} -submodule generated by $\{[K]f - e, [K] - f, \{R_a^1, R_a^2\}_{a \in A}\}$. Finally, define $S_K \in H$

by the following formula, where the summation is taken over all double points v of $\pi(K)$:

$$S_K = \sum_v ([\mu_v^+] - [\mu_v^-]). \quad (10)$$

Theorem 4.0.7. *S_K is an isotopy invariant of the knot K .*

For the proof of Theorem 4.0.7 see Subsection 4.0.7.

We introduce a similar invariant in the case where N is oriented and F is non-orientable.

Definition 4.0.8 (of \tilde{S}_K). Let N be an oriented Seifert fibered manifold with a non-orientable orbit space F . Let $\pi : N \rightarrow F$ be the corresponding fibration and $K \subset N$ an oriented knot in general position with respect to π . Assume also that K does not intersect the exceptional fibers. For each double point v of $\pi(K)$ we split K into $\mu_1^+(v)$, $\mu_1^-(v)$, $\mu_2^+(v)$, and $\mu_2^-(v)$ as in 3.4.1. The element $([\mu_1^+(v)] - [\mu_1^-(v)] + [\mu_2^+(v)] - [\mu_2^-(v)]) \in \mathbb{Z}[H_1(N)]$ is well defined.

Denote by f the homology class of a regular fiber oriented in some way. Note that $f^2 = e$, so the orientation we use to define f does not matter. For a cone point a denote by f_a the homology class of the fiber $\pi^{-1}(a)$ oriented in some way.

For $a \in A$ of type (μ_a, ν_a) set $N_1(a) = \{k \in \{1, \dots, \mu_a\} \mid \frac{2\pi k \nu_a}{\mu_a} \bmod 2\pi \in (0, \pi]\}$, $N_2(a) = \{k \in \{1, \dots, \mu_a\} \mid \frac{2\pi k \nu_a}{\mu_a} \bmod 2\pi \in (0, \pi)\}$.

Define $R_a \in \mathbb{Z}[H_1(N)]$ by the following formula:

$$\begin{aligned} R_a = & \sum_{k \in N_1(a)} \left([K] f_a^{\mu_a - k} - f_a^k + f_a^{k - \mu_a} - [K] f_a^{-k} \right) \\ & - \sum_{k \in N_2(a)} \left(f_a^{\mu_a - k} - [K] f_a^k + [K] f_a^{k - \mu_a} - f_a^{-k} \right) \quad (11) \end{aligned}$$

Let \tilde{H} be the quotient of $\mathbb{Z}[H_1(N)]$ (viewed as a \mathbb{Z} -module) by the free \mathbb{Z} -submodule generated by $\{(e - f)([K] + e), \{R_a\}_{a \in A}\}$.

One can prove that under the change of the orientation of $\pi^{-1}(a)$ (used to define f_a) R_a goes to $-R_a$. Thus \tilde{H} is well defined. To show this, one verifies that if μ_a is odd, then $N_1(a) = N_2(a)$. Under this change each term from the first sum (used to define R_a) goes to minus the corresponding term from the second sum and vice versa. (Note that $f^2 = e$.) If $\mu_a = 2l$ is even, then $N_1(a) \setminus \{l\} = N_2(a)$. Under this change each term with $k \in N_1(a) \setminus \{l\}$ goes to minus the corresponding term with $k \in N_2(a)$ and vice versa. The term in the first sum that corresponds to $k = l$ goes to minus itself.

Finally define $\tilde{S}_K \in \tilde{H}$ as the sum over all double points v of $\pi(K)$:

$$\tilde{S}_K = \sum_v \left([\mu_1^+(v)] - [\mu_1^-(v)] + [\mu_2^+(v)] - [\mu_2^-(v)] \right). \quad (12)$$

Theorem 4.0.9. *\tilde{S}_K is an isotopy invariant of K .*

The proof is a straightforward generalization of the proof of Theorem 4.0.7.

4.0.10. One can easily verify that S_K and \tilde{S}_K satisfy relations similar to (3). Hence both of them are Vassiliev invariants of order one (see 3.2.3).

5. WAVE FRONTS ON SURFACES

5.1. Definitions. Let F be a two-dimensional manifold. A *contact element* at a point in F is a one-dimensional vector subspace of the tangent plane. This subspace divides the tangent plane into two half-planes. A choice of one of them is called a *coorientation* of a contact element. The space of all cooriented contact elements of F is a spherical cotangent bundle ST^*F . We will also denote it by N . It is an S^1 -fibration over F . The natural contact structure on ST^*F is a distribution of hyperplanes given by the condition that a velocity vector of an incidence point of a contact element belongs to the element. A *Legendrian* curve λ in N is an immersion of S^1 into N such that for each $p \in S^1$ the velocity vector of λ at $\lambda(p)$ lies in the contact plane. The naturally cooriented projection $L \subset F$ of a Legendrian curve $\lambda \subset N$ is called *the wave front* of λ . A cooriented wave front may be uniquely lifted to a Legendrian curve $\lambda \subset N$ by taking a coorienting normal direction as a contact element at each point of the front. A wave front is said to be generic if it is an immersion everywhere except a finite number of points, where it has cusp singularities, and all multiple points are double points with transversal self-intersection. A cusp is the projection of a point where the corresponding Legendrian curve is tangent to the fiber of the bundle.

5.2. Shadows of wave fronts.

5.2.1. For any surface F the space ST^*F is canonically oriented. The orientation is constructed as follows. For a point $x \in F$ fix an orientation of $T_x F$. It induces an orientation of the fiber over x . These two orientations determine an orientation of three dimensional planes tangent to the points of the fiber over x . A straightforward verification shows that this orientation is independent on the orientation of $T_x F$ we choose. Hence the orientation of ST^*F is well defined.

Thus for oriented F the shadow of a generic knot in ST^*F is well defined (see 2.1 and 2.2.10). Theorem 5.2.3 describes the shadow of a Legendrian lifting of a generic cooriented wave front $L \subset F$.

Definition 5.2.2. Let X be a connected component of $F \setminus L$. We denote by $\chi \text{Int}(X)$ the Euler characteristic of $\text{Int}(X)$, by C_X^i the number of cusps in the boundary of the region X pointing inside X (as in Figure 8a), by C_X^o the number of cusps in the boundary of X pointing outside (as in Figure 8b), and by V_X the number of corners of X where locally the picture looks in one the two ways shown in Figure 8c. It can happen that a cusp point is pointing both inside and outside of X . In this case it contributes both in C_X^i and in C_X^o . If the corner of the type shown in Figure 8c enters twice in ∂X , then it should be counted twice.

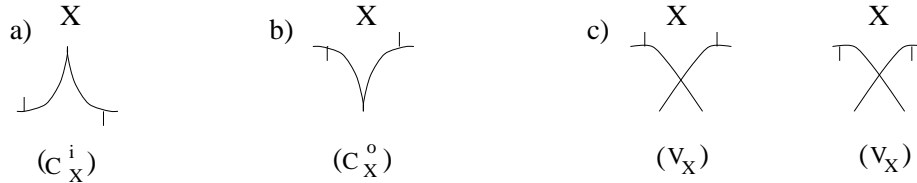


FIGURE 8.

Theorem 5.2.3. *Let F be an oriented surface and L a generic cooriented wave front on F corresponding to a Legendrian curve λ . There exists a small deformation of λ in the class of all smooth (not only Legendrian) curves such that the resulting curve is generic with respect to the projection, and the shadow of this curve can be constructed in the following way. We replace a small neighborhood of each cusp of L with a smooth simple arc. The gleam of an arbitrary region X that has a compact closure and does not contain boundary components of F is calculated by the following formula:*

$$\text{gl}_X = \chi \text{Int}(X) + \frac{1}{2}(C_X^i - C_X^o - V_X). \quad (13)$$

For the proof of Theorem 5.2.3 see Subsection 7.5.

Remark . The surface F in the statement of Theorem 5.2.3 is not assumed to be compact.

Note that as we mentioned in 2.2.10, the gleam of a region X that does not have compact closure or contains boundary components is not well defined unless we fix a section over all ends of X and components of ∂F in X .

This theorem first appeared in [9]. A similar result was independently obtained by Polyak [6].

5.2.4. A self-tangency point p of a wave front is said to be a point of a *dangerous self-tangency* if the coorienting normals of the two branches coincide at p (see Figure 9). Dangerous self-tangency points correspond to self-intersection of the Legendrian curve. Hence a generic deformation of the front L not involving *dangerous* self-tangencies corresponds to an isotopy of the Legendrian knot λ .



FIGURE 9.

Any generic deformation of a wave front L corresponding to an isotopy in the class of the Legendrian knots can be split into a sequence of modifications shown in Figure 10. The construction of Theorem 5.2.3 transforms these generic modifications of wave fronts to shadow moves: Ia and Ib in Figure 10 are transformed to the \bar{S}_1 move for shadow diagrams, IIa, IIb, II'a, II'b, II'c, and II'd are transformed to the S_2 move, finally IIIa and IIIb are transformed to S_3 and \bar{S}_3 respectively.

5.2.5. Thus for the Legendrian lifting of a wave front we are able to calculate all invariants that we can calculate for shadows. This includes the analogue of the linking number for the fronts on \mathbb{R}^2 (see [10]), the second order Vassiliev invariant (see [7]), and quantum state sums (see [10]).

5.3. Invariants of wave fronts on surfaces. In particular, the S_K invariant gives rise to an invariant of a generic wave front. This invariant appears to be related to the formula for the Bennequin invariant of a wave front introduced by Polyak in [5].

Let us recall the corresponding results and definitions of [5].

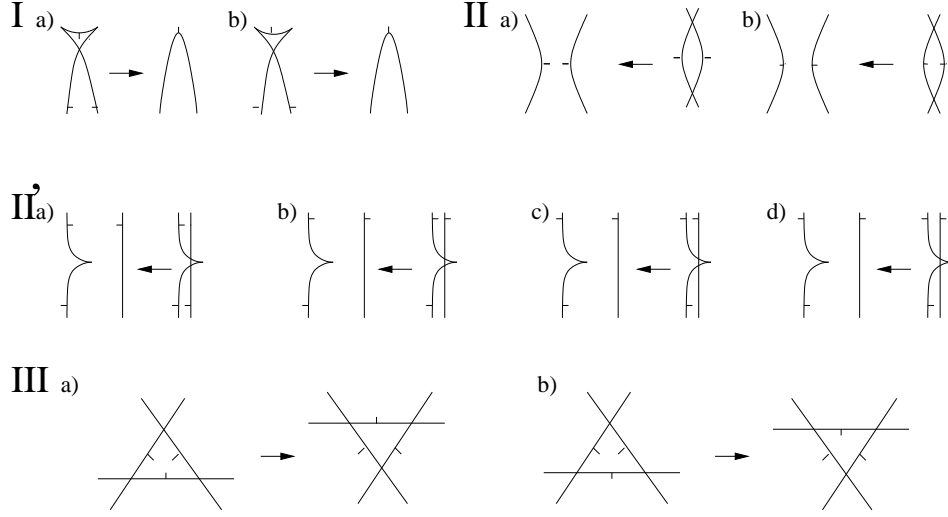


FIGURE 10.

Let L be a generic cooriented oriented wave front on an oriented surface F . A branch of a wave front is said to be positive (resp. negative) if the frame of coorienting and orienting vectors defines positive (resp. negative) orientation of the surface F . Define the *sign* $\epsilon(v)$ of a double point v of L to be $+1$ if the signs of both branches of the front intersecting at v coincide and -1 otherwise. Similarly we assign a positive (resp. negative) sign to a cusp point if the coorienting vector turns in a positive (resp. negative) direction while traversing a small neighborhood of the cusp point along the orientation. We denote half of the number of positive and negative cusp points by C^+ and C^- respectively.

Let v be a double point of L . The orientations of F and L allow one to distinguish the two wave fronts L_v^+ and L_v^- obtained by splitting of L in v with respect to orientation and coorientation (see Figures 11a.1 and 11b.1). (Locally one of the two fronts lies to the left and another to the right of v .)

For a Legendrian curve λ in $ST^*\mathbb{R}^2$ denote by $l(\lambda)$ its Bennequin invariant described in the works of Tabachnikov [8] and Arnold [3] with the sign convention of [3] and [5].

Theorem 5.3.1 (Polyak [5]). *Let L be a generic oriented cooriented wave front on \mathbb{R}^2 and λ the corresponding Legendrian curve. Denote by $\text{ind}(L)$ the degree of the mapping taking a point $p \in S^1$ to the point of S^1 corresponding to the direction of the coorienting normal of L at $L(p)$. Define S as the following sum over all double points of L :*

$$S = \sum_v (\text{ind}(L_v^+) - \text{ind}(L_v^-) - \epsilon(v)). \quad (14)$$

Then

$$l(\lambda) = S + (1 - \text{ind}(L))C^+ + (\text{ind}(L) + 1)C^- + \text{ind}(L)^2. \quad (15)$$

In [5] it is shown that the Bennequin invariant of a wave front on the \mathbb{R}^2 plane admits quantization. Consider a formal quantum parameter q . Recall that for

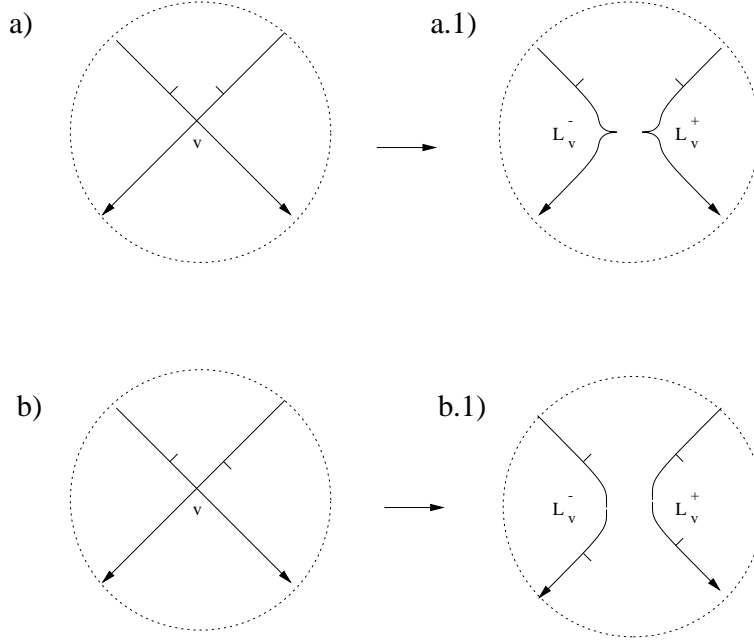


FIGURE 11.

any $n \in \mathbb{Z}$ the corresponding quantum number $[n]_q \in \mathbb{Z}[q, q^{-1}]$ is a finite Laurent polynomial in q defined by

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \quad (16)$$

Substituting quantum integers instead of integers in 5.3.1 we get the following theorem.

Theorem 5.3.2 (Polyak [5]). *Let L be a generic cooriented oriented wave front on \mathbb{R}^2 and λ the corresponding Legendrian curve. Define S_q by the following formula, where the sum is taken over the set of all double points of L :*

$$S_q = \sum_v [\text{ind}(L_v^+) - \text{ind}(L_v^-) - \epsilon(v)]_q. \quad (17)$$

Put

$$l_q(L) = S_q + [1 - \text{ind}(L)]_q C^+ + [\text{ind}(L) + 1]_q C^- + [\text{ind}(L)]_q \text{ind}(L). \quad (18)$$

Then $l_q(\lambda) = l_q(L) \in \frac{1}{2}\mathbb{Z}[q, q^{-1}]$ is invariant under isotopy in the class of the Legendrian knots.

The $l_q(\lambda)$ invariant can be expressed [2] in terms of the partial linking polynomial of a generic cooriented oriented wave front introduced by Aicardi [1].

The reason why this invariant takes values in $\frac{1}{2}\mathbb{Z}[q, q^{-1}]$ and not in $\mathbb{Z}[q, q^{-1}]$ is that the number of positive (or negative) cusps can be odd. This makes C^+ (C^-) a half-integer.

Let λ_v^ϵ with $\epsilon = \pm$ be the Legendrian lifting of the front L_v^ϵ . Let $f \in H_1(ST^*F)$ be the homology class of a positively oriented fiber.

Theorem 5.3.3 (Polyak [5]). *Let L be a generic oriented cooriented wave front on an oriented surface F . Let λ be the corresponding Legendrian curve. Define $l_F(\lambda) \in H_1(ST^*F, \frac{1}{2}\mathbb{Z})$ by the following formula:*

$$l_F(\lambda) = \left(\prod_v [\lambda_v^+] [\lambda_v^-]^{-1} f^{-\epsilon(v)} \right) (f[\lambda]^{-1})^{C^+} ([\lambda]f)^{C^-} \quad (19)$$

(We use the multiplicative notation for the group operation in $H_1(ST^*F)$.)

Then $l_F(\lambda)$ is invariant under isotopy in the class of the Legendrian knots.

The proof is straightforward. One verifies that $l_F(\lambda)$ is invariant under all oriented versions of non-dangerous self-tangency, triple point, cusp crossing, and cusp birth moves of the wave front.

In [5] this invariant is denoted by $I_\Sigma^+(\lambda)$ and, in a sense, it appears to be a natural generalization of Arnold's J^+ invariant [3] to the case of an oriented cooriented wave front on an oriented surface.

Note that in the situation of Theorem 5.3.1 the indices of all the fronts involved are the images of the homology classes of their Legendrian liftings under the natural identification of $H_1(ST^*\mathbb{R}^2)$ with \mathbb{Z} . If one replaces everywhere in 5.3.1 indices with the corresponding homology classes and puts f instead of 1, then the only difference between the two formulas is the term $\text{ind}^2(L)$. (One has to remember that we use the multiplicative notation for the operation in $H_1(ST^*F)$.)

5.3.4. The splitting of a Legendrian knot K into μ_v^+ and μ_v^- (see 3.1.1) can be done up to an isotopy in the class of the Legendrian knots. Although this can be done in many ways, there exists the simplest way. The projections \tilde{L}_v^+ and \tilde{L}_v^- of the Legendrian curves created by the splitting are shown in Figure 12. (This fact follows from Theorem 5.2.3.)

Let $\tilde{\lambda}_v^\epsilon$ with $\epsilon = \pm$ be the Legendrian lifting of the front \tilde{L}_v^ϵ .

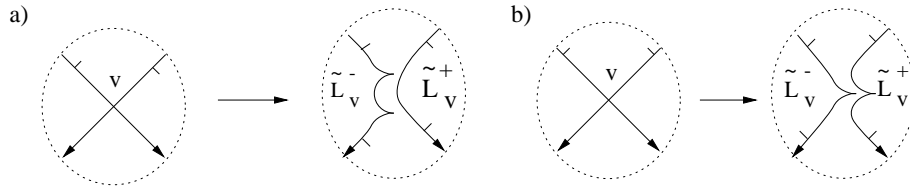


FIGURE 12.

Theorem 5.3.5. *Let L be a generic oriented cooriented wave front on an oriented surface F . Let λ be the corresponding Legendrian curve. Define $S(\lambda) \in \frac{1}{2}\mathbb{Z}[H_1(ST^*F)]$ by the following formula:*

$$S(\lambda) = \sum_v \left([\tilde{\lambda}_v^+] - [\tilde{\lambda}_v^-] \right) + (f - [\lambda])C^+ + ([\lambda]f - e)C^-. \quad (20)$$

Then $S(\lambda)$ is invariant under isotopy in the class of the Legendrian knots.

The proof is straightforward. One verifies that $S(\lambda)$ is indeed invariant under all oriented versions of non-dangerous self-tangency, triple point, cusp crossing, and cusp birth moves of the wave front.

5.3.6. By taking the free homotopy classes of $\tilde{\lambda}_v^+$ and $\tilde{\lambda}_v^-$ instead of the homology classes one obtains a different version of the $S(\lambda)$ invariant. It takes values in the group of formal half-integer linear combinations of the free homotopy classes of oriented curves in ST^*F . In this case the terms $[\lambda]$ and f in (20) should be substituted with the free homotopy classes of λ and of a positively oriented fiber respectively. The terms $[\lambda]f$ and e in (20) should be substituted with the free homotopy classes of λ with a positive fiber added to it and the class of a contractible curve respectively. Note that f lies in the center of $\pi_1(ST^*F)$, so that the class of λ with a fiber added to it is well defined.

A straightforward verification shows that this version of $S(\lambda)$ is also invariant under isotopy in the class of the Legendrian knots.

Theorem 5.3.7. *Let L be a generic oriented cooriented wave front on an oriented surface F . Let λ be the corresponding Legendrian curve. Let $S(\lambda)$ and $l_F(\lambda)$ be the invariants introduced in 5.3.5 and 5.3.3 respectively. Let*

$$\text{pr} : \frac{1}{2}\mathbb{Z}[H_1(ST^*F)] \rightarrow H_1(ST^*F, \frac{1}{2}\mathbb{Z}) \quad (21)$$

*be the mapping defined as follows: for any $n_i \in \frac{1}{2}\mathbb{Z}$ and $g_i \in H_1(ST^*F)$,*

$$\sum n_i g_i \mapsto \prod g_i^{n_i}. \quad (22)$$

Then $\text{pr}(S(\lambda)) = l_F(\lambda)$.

The proof is straightforward: one must verify that

$$[\lambda_v^+][\lambda_v^-]^{-1} f^{-\epsilon(v)} = [\tilde{\lambda}_v^+][\tilde{\lambda}_v^-]^{-1} \text{ in } H_1(ST^*F). \quad (23)$$

(Recall that we use a multiplicative notation for the group operation in $H_1(ST^*F)$.)

This means that $S_F(\lambda)$ is a refinement of Polyak's invariant $l_F(\lambda)$.

5.3.8. One can verify that there is a unique linear combination $\sum_{m \in \mathbb{Z}} n_m [m]_q = l_q(\lambda)$ with n_m being non-negative half-integers such that $n_0 = 0$, and if $n_m > 0$, then $n_{-m} = 0$. To prove this one must verify that $\{\frac{1}{2}[n]_q | 0 < n\}$ is a basis for the \mathbb{Z} -submodule of $\frac{1}{2}\mathbb{Z}[q, q^{-1}]$ generated by the quantum numbers and use the identity $n[m]_q = -n[-m]_q$.

The following theorem shows that if $L \subset \mathbb{R}^2$, then $S(\lambda)$ and Polyak's quantization $l_q(\lambda)$ (see 5.3.2) of the Bennequin invariant can be explicitly expressed in terms of each other.

Theorem 5.3.9. *Let $f \in H_1(ST^*\mathbb{R}^2)$ be the class of a positively oriented fiber. Let L be a generic oriented cooriented wave front on \mathbb{R}^2 , λ the corresponding Legendrian curve, and f^h the homology class realized by it. Let $l_q(\lambda) = [h]_q h = \sum_{m \in \mathbb{Z}} n_m [m]_q$ be written in the form described in 5.3.8 and $S(\lambda) = \sum_{l \in \mathbb{Z}} k_l f^l$. Then*

$$l_q(\lambda) = [h]_q h + \sum_{k_l > 0} k_l [2l - h - 1]_q, \quad (24)$$

and

$$S(\lambda) = \sum_{n_m > 0} n_m (f^{\frac{h+1+m}{2}} - f^{\frac{h+1-m}{2}}). \quad (25)$$

For the proof of Theorem 5.3.9 see Subsection 7.6.

One can show that for $n_m > 0$ both $\frac{h+1+m}{2}$ and $\frac{h+1-m}{2}$ are integers, so that the sum (25) takes values in $\frac{1}{2}\mathbb{Z}[H_1(ST^*\mathbb{R}^2)]$.

Note that the $l_q(\lambda)$ invariant was defined only for fronts on the plane \mathbb{R}^2 . Thus $S(\lambda)$ is, in a sense, a generalization of Polyak's $l_q(\lambda)$ to the case of wave fronts on an arbitrary oriented surface F .

5.3.10. The splitting of the Legendrian knot K into $\mu_1^+(v)$, $\mu_1^-(v)$, $\mu_2^+(v)$, and $\mu_2^-(v)$ (which was used to define $\tilde{S}(K)$, see 3.4.1) can be done up to an isotopy in the class of the Legendrian knots. Although this can be done in many ways, there is the simplest one. The projections $\tilde{L}_1^+(v)$, $\tilde{L}_1^-(v)$, $\tilde{L}_2^+(v)$, and $\tilde{L}_2^-(v)$ are shown in Figure 13. (This fact follows from Theorem 5.2.3.)

This allows us to introduce an invariant similar to $S(\lambda)$ for generic oriented cooriented wave fronts on a non-orientable surface F in the following way.

Let L be a generic wave front on a non-orientable surface F . Let v be a double point of L . Fix some orientation of a small neighborhood of v in F . The orientations of the neighborhood and L allow one to distinguish the wave fronts L_1^+ , L_1^- , L_2^+ , and L_2^- obtained by the two splittings of L with respect to the orientation and coorientation (see Figure 13). Locally the fronts with the upper indices plus and minus are located respectively to the right and to the left of v . To each double point v of L we associate an element $([\tilde{\lambda}_1^+(v)] - [\tilde{\lambda}_1^-(v)] + [\tilde{\lambda}_2^+(v)] - [\tilde{\lambda}_2^-(v)]) \in \mathbb{Z}[H_1(ST^*(F))]$. Here we denote by λ s the Legendrian curves corresponding to the wave fronts appearing under the splitting. Clearly this element does not depend on the orientation of the neighborhood of v we have chosen.

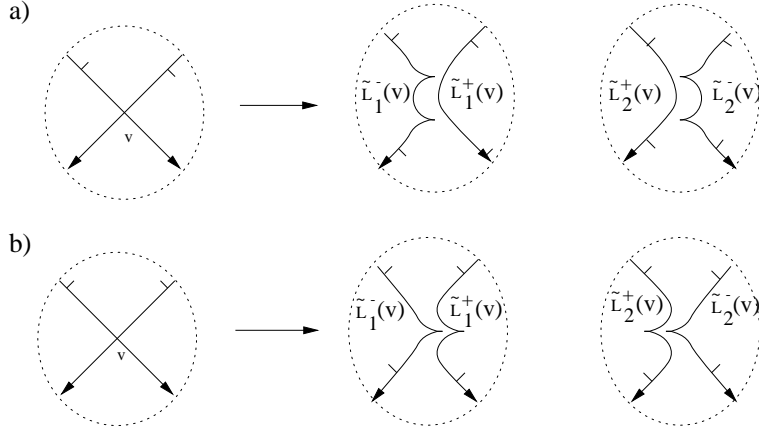


FIGURE 13.

For a wave front L let C be half of the number of cusps of L . Denote by f the homology class of the fiber of ST^*F oriented in some way. Note that $f^2 = e$, so it does not matter which orientation of the fiber we use to define f .

Theorem 5.3.11. *Let L be a generic cooriented oriented wave front on a non-orientable surface F and λ the corresponding Legendrian curve. Define $\tilde{S}(\lambda) \in$*

$\frac{1}{2}\mathbb{Z}[H_1(ST^*(F))]$ by the following formula, where the summation is taken over the set of all double points of L :

$$\tilde{S}(\lambda) = \sum_v \left([\tilde{\lambda}_1^+(v)] - [\tilde{\lambda}_1^-(v)] + [\tilde{\lambda}_2^+(v)] - [\tilde{\lambda}_2^-(v)] \right) + C([\lambda]f - e + f - [\lambda]). \quad (26)$$

Then $\tilde{S}(\lambda)$ is invariant under isotopy in the class of the Legendrian knots.

The proof is straightforward. One verifies that $\tilde{S}(\lambda)$ is indeed invariant under all oriented versions of non-dangerous self-tangency, triple point passing, cusp crossing, and cusp birth moves of the wave front.

The reason we have $\tilde{S}(\lambda) \in \frac{1}{2}\mathbb{Z}[H_1(ST^*F)]$ is that if L is an orientation reversing curve, then the number of cusps of L is odd. In this case C is a half-integer.

6. WAVE FRONTS ON ORBIFOLDS

6.1. Definitions.

Definition 6.1.1. An *orbifold* is a surface F with the additional structure consisting of:

- 1) set $A \subset F$;
- 2) smooth structure on $F \setminus A$;
- 3) set of homeomorphisms ϕ_a of neighborhoods U_a of a in F onto \mathbb{R}^2/G_a such that $\phi_a(a) = 0$ and $\phi_a|_{U_a \setminus a}$ is a diffeomorphism. Here $G_a = \{e^{\frac{2\pi k}{\mu_a}} | k \in \{1, \dots, \mu_a\}\}$ is a group acting on $\mathbb{R}^2 = \mathbb{C}$ by multiplication. ($\mu_a \in \mathbb{Z}$ is assumed to be positive.)

The points $a \in A$ are called *cone points*.

The action of G on \mathbb{R}^2 induces the action of G on $ST^*\mathbb{R}^2$. This makes $ST^*\mathbb{R}^2/G$ a Seifert fibration over \mathbb{R}^2/G . Gluing together the pieces over neighborhoods of F we obtain a Seifert fibration $\pi : N \rightarrow F$. The fiber over a cone point a is an exceptional fiber of type $(\mu_a, -1)$ (see 4.0.5).

The natural contact structure on $ST^*\mathbb{R}^2$ is invariant under the induced action of G . Since G acts freely on $ST^*\mathbb{R}^2$, this implies that N has an induced contact structure. As before, the naturally cooriented projection $L \subset F$ of a generic Legendrian curve λ is called *the front of λ* .

6.2. Invariants for fronts on orbifolds. For oriented F we construct an invariant similar to $S(\lambda)$. It corresponds to the S_K invariant of a knot in a Seifert fibered space. For non-orientable surface F we construct an analogue of $\tilde{S}(\lambda)$. It corresponds to the \tilde{S}_K invariant of a knot in a Seifert fibered space.

Note that any surface F can be viewed as an orbifold without cone points. This justifies the notation below.

Let F be an oriented surface. The orientation of F induces an orientation of all fibers. Denote by f the homology class of a positively oriented fiber. For a cone point a denote by f_a the homology class of a positively oriented fiber $\pi^{-1}(a)$. For a generic oriented cooriented wave front $L \subset F$ denote by C^+ (resp. C^-) half of the number of positive (resp. negative) cusps of L . Note that for a double point v of a generic front L the splitting into \tilde{L}_v^+ and \tilde{L}_v^- is well defined. The corresponding Legendrian curves $\tilde{\lambda}_v^+$ and $\tilde{\lambda}_v^-$ in N are also well defined.

For $a \in A$ of type $(\mu_a, -1)$ put $N_1(a) = \{k \in \{1, \dots, \mu_a\} | \frac{-2k\pi}{\mu_a} \bmod 2\pi \in (0, \pi]\}$, $N_2(a) = \{k \in \{1, \dots, \mu_a\} | \frac{2k\pi}{\mu_a} \bmod 2\pi \in (0, \pi)\}$. Define $R_a^1, R_a^2 \in \mathbb{Z}[H_1(N)]$ by the following formulas:

$$R_a^1 = \sum_{k \in N_1(a)} ([\lambda] f_a^{\mu_a - k} - f_a^k) - \sum_{k \in N_2(a)} (f_a^{\mu_a - k} - [\lambda] f_a^k), \quad (27)$$

$$R_a^2 = \sum_{k \in N_1(a)} (f_a^{\mu_a - k} - [\lambda] f_a^k) - \sum_{k \in N_2(a)} ([\lambda] f_a^{\mu_a - k} - f_a^k). \quad (28)$$

Set J to be the quotient of $\frac{1}{2}\mathbb{Z}[H_1(N)]$ by the free Abelian subgroup generated by $\left\{ \frac{1}{2}R_1(a), \frac{1}{2}R_2(a) \right\}_{a \in A}$.

Theorem 6.2.1. *Let L be a generic cooriented oriented wave front on F and λ the corresponding Legendrian curve.*

Then $S(\lambda) \in J$ defined by the sum over all double points of L ,

$$S(\lambda) = \sum \left([\tilde{\lambda}^+(v)] - [\tilde{\lambda}^-(v)] \right) + (f - [\lambda])C^+ + ([\lambda]f - e)C^-, \quad (29)$$

is invariant under isotopy in the class of the Legendrian knots.

For the proof of Theorem 6.2.1 see Subsection 7.7.

Let F be a non-orientable surface. Denote by f the homology class of a regular fiber oriented in some way. Note that $f^2 = e$, so the orientation we use to define f does not matter. For a cone point a denote by f_a the homology class of the fiber $\pi^{-1}(a)$ oriented in some way. For a generic oriented cooriented wave front $L \subset F$ denote by C half of the number of cusps of L . Note that for a double point v of a generic front L the element $([\tilde{\lambda}_1^+(v)] - [\tilde{\lambda}_1^-(v)] + [\tilde{\lambda}_2^+(v)] - [\tilde{\lambda}_2^-(v)]) \in \mathbb{Z}[H_1(N)]$ used to introduce $\tilde{S}(\lambda)$ is well defined.

For $a \in A$ of type $(\mu_a, -1)$ put $N_1(a) = \{k \in \{1, \dots, \mu_a\} \mid \frac{-2k\pi}{\mu_a} \bmod 2\pi \in (0, \pi]\}$, and $N_2(a) = \{k \in \{1, \dots, \mu_a\} \mid \frac{-2k\pi}{\mu_a} \bmod 2\pi \in (0, \pi)\}$. Define $R_a \in \mathbb{Z}[H_1(N)]$ by the following formula:

$$R_a = \sum_{k \in N_1(a)} \left([\lambda] f_a^{\mu_a - k} - f_a^k + f_a^{k - \mu_a} - [\lambda] f_a^{-k} \right) - \sum_{k \in N_2(a)} \left(f_a^{\mu_a - k} - [\lambda] f_a^k + [\lambda] f_a^{k - \mu_a} - f_a^{-k} \right). \quad (30)$$

Put \tilde{J} to be the quotient of $\frac{1}{2}\mathbb{Z}[H_1(N)]$ by a free Abelian subgroup generated by $\left\{ \frac{1}{2}R_a \right\}_{a \in A}$.

Similarly to 4.0.8, one can prove that under the change of the orientation of $\pi^{-1}(a)$ (used to define f_a) R_a goes to $-R_a$. Thus \tilde{J} is well defined.

Theorem 6.2.2. *Let L be a generic cooriented oriented wave front on F and λ the corresponding Legendrian curve.*

Then $\tilde{S}(\lambda) \in \tilde{J}$ defined by the summation over all double points of L ,

$$\tilde{S}(\lambda) = \sum \left([\tilde{\lambda}_1^+(v)] - [\tilde{\lambda}_1^-(v)] + [\tilde{\lambda}_2^+(v)] - [\tilde{\lambda}_2^-(v)] \right) + ([\tilde{\lambda}]f - e + f - [\lambda])C, \quad (31)$$

is invariant under isotopy in the class of the Legendrian knots.

The proof is a straightforward generalization of the proof of Theorem 6.2.1.

7. PROOFS

7.1. Proof of Theorem 3.1.3. To prove the theorem it suffices to show that S_K is invariant under the elementary isotopies. They project to: a death of a double point, cancellation of two double points, and passing through a triple point.

To prove the invariance, we fix a homeomorphic to a closed disk part P of F containing the projection of one of the elementary isotopies. Fix a section over the boundary of P such that the points of $K \cap \pi^{-1}(\partial P)$ belong to the section. Inside P we can construct the Turaev shadow (see 2.2.10). The action of $H_1(\text{Int } P) = e$ on the set of isotopy types of links is trivial (see 2.2.8). Thus the part of K can be reconstructed in the unique way from the shadow over P . In particular, one can compare the homology classes of the curves created by splitting at a double point inside P . Hence to prove the theorem, it suffices to verify the invariance under the oriented versions of the moves S_1 , S_2 , and S_3 .

There are two versions of the oriented move S_1 shown in Figures 14a and 14b.

For Figure 14a the term $[\mu_v^+]$ appears to be equal to f . From 3.1.2 we know that $[\mu_v^+][\mu_v^-] = [K]f$, so that $[\mu_v^-] = [K]$. Hence $[\mu_v^+] - [\mu_v^-] = f - [K]$ and is equal to zero in H . In the same way we verify that $[\mu_v^+] - [\mu_v^-]$ (for v shown in Figure 14b) is equal to $[K]f - e$. It is also zero in H . The summands corresponding to other double points do not change under this move, since it does not change the homology classes of the knots created by the splittings.

There are three oriented versions of the S_2 move. We show that S_K does not change under one of them. The proof for the other two is the same or easier. We choose the version corresponding to the upper part of Figure 15. The summands corresponding to the double points not in this figure are preserved under the move, since it does not change the homology classes of the corresponding knots. So it suffices to show that the terms produced by the double points u and v in this figure cancel out. Note that the shadow μ_v^- is transformed to μ_u^+ by the \tilde{S}_1 move. It is known that \tilde{S}_1 can be expressed in terms of S_1 , S_2 , and S_3 , thus it also does not change the homology classes of the knots created by the splittings. Hence $[\mu_u^+]$ and $[\mu_v^-]$ cancel out. In the same way one proves that $[\mu_u^-]$ and $[\mu_v^+]$ also cancel out.

There are two oriented versions of the S_3 move: S'_3 and S''_3 , shown in Figures 16a and 16b respectively. The S''_3 move can be expressed in terms of S'_3 and oriented versions of S_2 and S_2^{-1} . To prove this we use Figure 17. There are two ways to get from Figure 17a to Figure 17b. One is to apply S''_3 . Another way is to apply three times the oriented version of S_2 to obtain Figure 17c, then apply S'_3 to get Figure 17d, and finally use three times the oriented version of S_2^{-1} to end up at Figure 17b.

Thus it suffices to verify the invariance under S'_3 . The terms corresponding to the double points not in Figures 18a and 18b are preserved for the same reasons as above. The terms coming from double points u in Figure 18a and u in Figure 18b are the same. This holds also for the v - and w -pairs of double points in these two figures. We prove this statement only for the u -pair of double points. For v - and w -pairs the proof is the same or simpler. There is only one possibility: either the dashed line belongs to both $\pi(\mu_u^+)$ in Figures 18a.1 and 18b.1 respectively or to both $\pi(\mu_u^-)$ in Figures 18a.2 and 18b.2 respectively. We choose the one to which it does not belong. Summing up gleams on each of the two sides of it we immediately see that the corresponding shadows are the same on both pictures. Thus the homology classes of the corresponding knots are equal. But $[\mu_u^+][\mu_u^-] = [K]f$ (see 3.1.2), thus

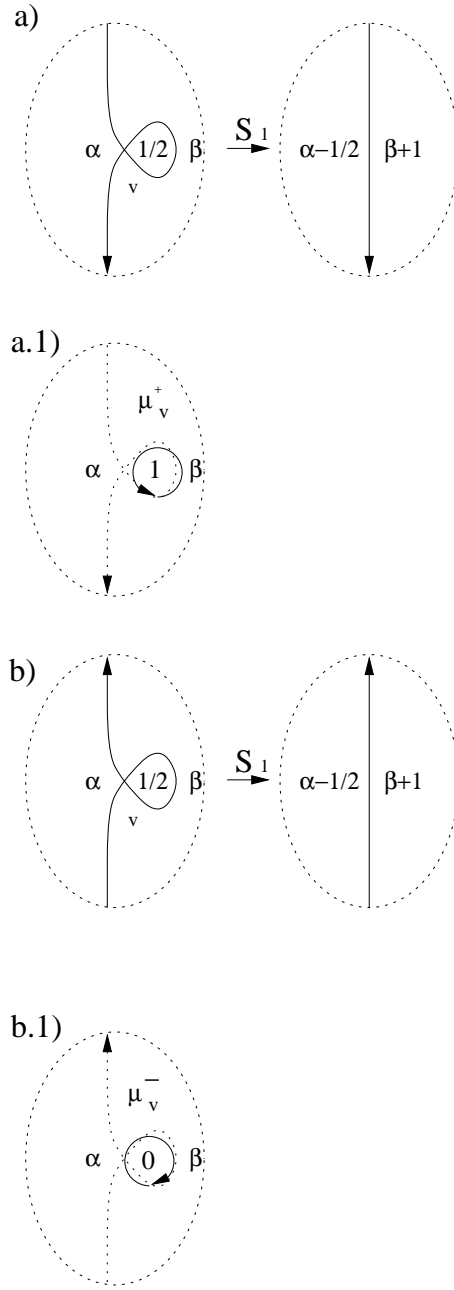


FIGURE 14.

the homology classes of the knots represented by the other shadows are also equal. This completes the proof of Theorem 3.1.3. \square

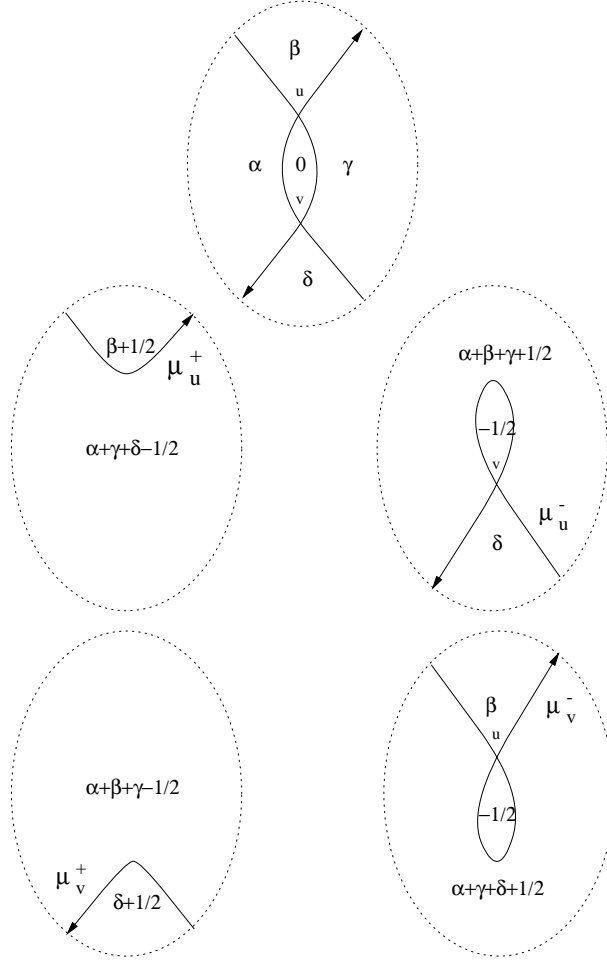


FIGURE 15.

7.2. Proof of Theorem 3.2.4. I: K' can be obtained from K by a sequence of isotopies and modifications along fibers. Isotopies do not change S . The modifications change S by elements of type (3). To complete the proof we use the identity $\xi_1 \xi_2 = [K]$.

II: We prove that for any $i \in H_1(N)$ there exist two knots K_1 and K_2 such that they represent the same free homotopy class as K ,

$$S_{K_1} = S_K + (f - e)([K]i^{-1} + i), \text{ and} \quad (32)$$

$$S_{K_2} = S_K - (f - e)([K]i^{-1} + i). \quad (33)$$

Clearly this would imply the second statement of the theorem.

Take $i \in H_1(N)$. Let K_i be an oriented knot in N such that $[K_i] = i$. The space N is oriented, hence the tubular neighborhood T_{K_i} of K_i is homeomorphic to an oriented solid torus T . Deform K_i , so that $K_i \cap T_{K_i}$ is a small arc (see Figure 19). Pull one part of the arc along K_i in T_{K_i} under the other part of the arc

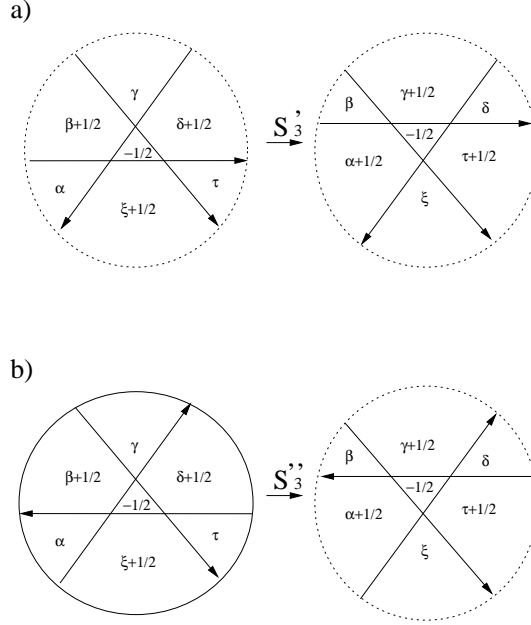


FIGURE 16.

(see Figure 20). This isotopy creates two new double points u and v of $\pi(K)$. (Since T_{K_i} may be knotted, it might happen that there are other new double points, but we do not need them for our construction.) Making a fiber modification along the part of $\pi^{-1}(u)$ that lies in T one obtains K_2 . Making a fiber modification along the part of $\pi^{-1}(v)$ that lies in T one obtains K_1 . This completes the proof of Theorem 3.2.4. \square

7.3. Proof of Theorem 3.3.3. It is easy to verify that any two shadows with the same projection can be transformed to each other by a sequence of fiber fusions. One can easily create a trivial knot with an ascending diagram such that its projection is any desired curve. This implies that any two shadows on \mathbb{R}^2 can be transformed to each other by a sequence of fiber fusions, movements S_1, S_2, S_3 , and their inverses. A straightforward verification shows that $\sigma(s(K))$ does not change under the moves S_1, S_2, S_3 , and their inverses. Under fiber fusions the homology class of the knot and the element σ change in the same way. To prove this, we use Figure 21, where Figure 21a shows the shadow before the application of the fiber fusion (that adds 1 to the homology class of the knot) and Figure 21b after. In this diagrams the indices of the regions are denoted by Latin letters. Now one easily verifies that σ also increases by one. Finally, for the trivial knot with a trivial shadow diagram its homology class and $\sigma(s(K))$ are both equal to 0. This completes the proof of Theorem 3.3.3. \square

7.4. Proof of Theorem 4.0.7. It suffices to show that S_K does not change under the elementary isotopies of the knot. Three of them correspond in the projection to: a birth of a small loop, passing through a point of self-tangency, and passing through a triple point. The fourth one is passing through an exceptional fiber.

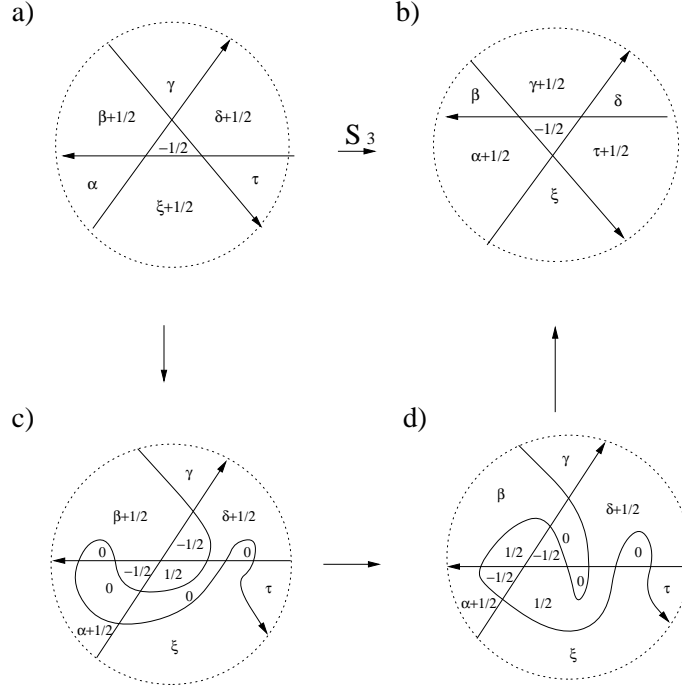


FIGURE 17.

From the proof of Theorem 3.1.3 one gets that S_K is invariant under the first three of the elementary isotopies described above. Thus it suffices to prove invariance under passing through an exceptional fiber a .

Let a be a singular fiber of type (μ_a, ν_a) (see 4.0.5). Let T_a be a neighborhood of a which is fiber-wise isomorphic to the standardly fibered solid torus with an exceptional fiber of type (μ_a, ν_a) .

We can assume that the move proceeds as follows. At the start K and T_a intersect along a curve lying in the meridional disk D of T_a . The part of K close to a in D is an arc C of a circle of a very large radius. This arc is symmetric with respect to the y axis passing through a in D . During the move this arc slides along the y axis through the fiber a (see Figure 22).

Clearly two points u and v of C after this move are in the same fiber if and only if they are symmetric with respect to the y axis, and the angle formed by v, a, u in D is less or equal to π and is equal to $\frac{2l\pi}{\mu_a}$ for some $l \in \{1, \dots, \mu_a\}$ (see Figure 22). They are in the same fiber before the move if and only if the angle formed by u, a, v in D is less than π and is equal to $\frac{2l\pi}{\mu_a}$ for some $l \in \{1, \dots, \mu_a\}$ (see Figure 22).

Consider a double point v of $\pi|_D(K)$ that appears after the move and corresponds to the angle $\frac{2l\pi}{\mu_a}$. There is a unique $k \in N_1(a)$ such that $\frac{2\pi\nu_a k}{\mu_a} \bmod 2\pi = \frac{2l\pi}{\mu}$. Note that to make the splitting of $[K]$ into $[\mu_v^+]$ and $[\mu_v^-]$ well defined, we do not need the two points of K projecting to v to be antipodal in $\pi^{-1}(v)$. This allows one to compare these homology classes with f_a . For the orientation of C shown in Figure 22 one verifies that connecting v to u along the orientation of the fiber we are adding k fibers f_a . (Note that the factorization we used to define the exceptionally

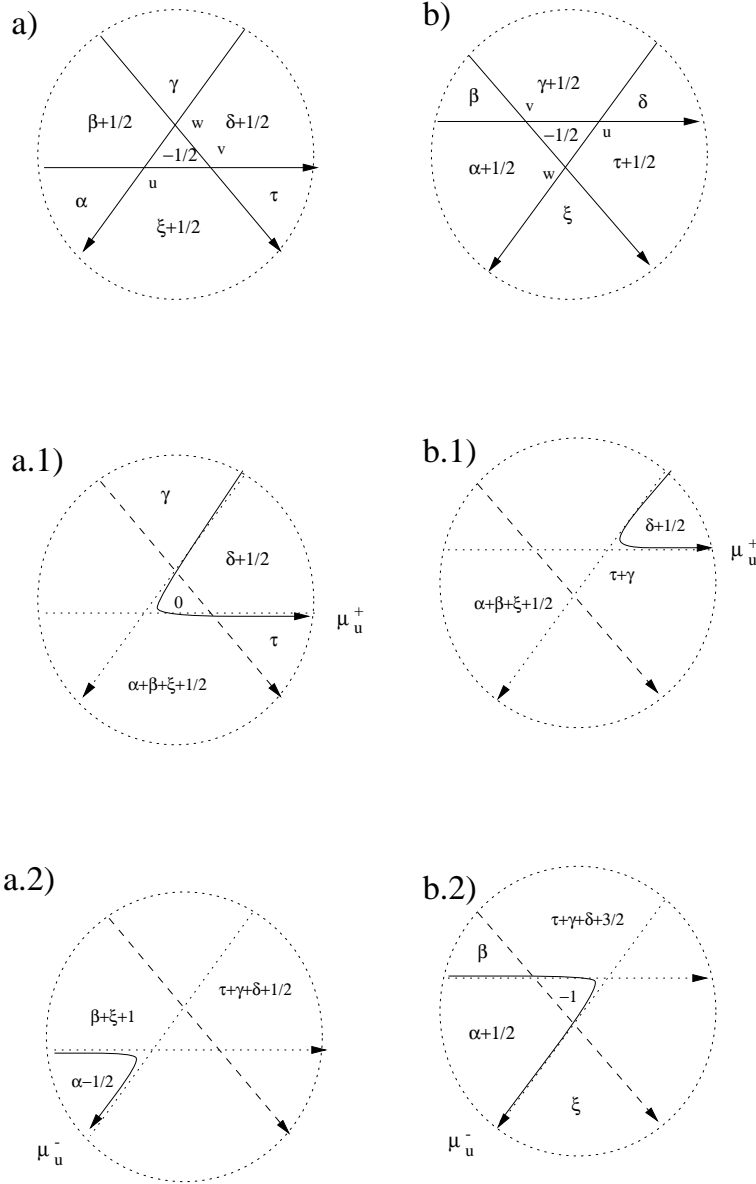


FIGURE 18.

fibered torus was $((r, \theta), 1) = ((r, \theta + \frac{2\pi\nu}{\mu}), 0)$. Thus $[\mu_v^-] = f_a^k$ (see Figure 23). From 3.1.2 we know that $[\mu_v^+][\mu_v^-] = [K]f$. Hence $[\mu_v^+] = [K]f_a^{\mu_a - k}$.

As above, to each double point v of $\pi|_D(K)$ before this move there corresponds $k \in N_2(a)$. For this double point $[\mu_v^+] = f_a^{\mu_a - k}$ and $[\mu_v^-] = [K]f_a^k$.

Summing up over the corresponding values of k we see that S_K changes by R_a^1 under this move. Recall that $R_a^1 = 0 \in H$. Thus S_K is invariant under the move.

For the other choice of the orientation of C the value of S_K changes by $R_a^2 = 0 \in H$.

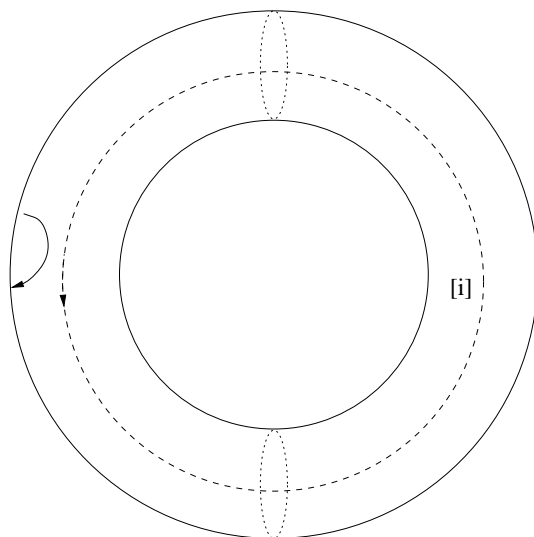


FIGURE 19.

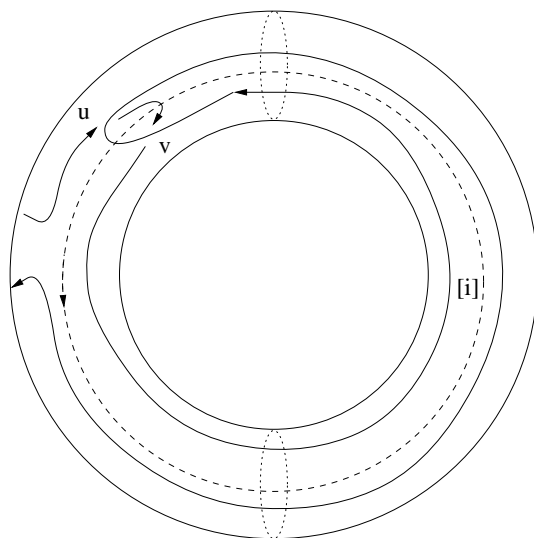


FIGURE 20.



FIGURE 21.

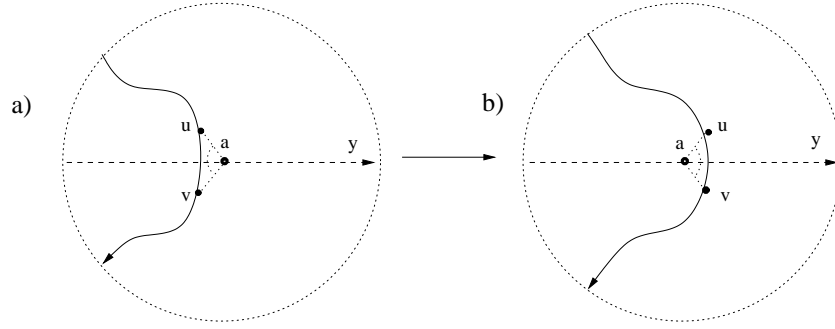


FIGURE 22.

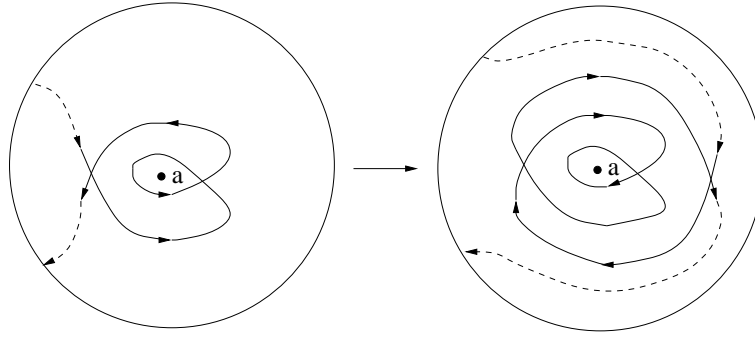


FIGURE 23.

Thus S_K is invariant under all elementary isotopies, and this proves Theorem 4.0.7. \square

7.5. Proof of Theorem 5.2.3. Deform L in the neighborhoods of all double points of L (see Figure 24), so that the two points of the Legendrian knot corresponding to the double point of L are antipodal in the fiber. After we make the quotient of

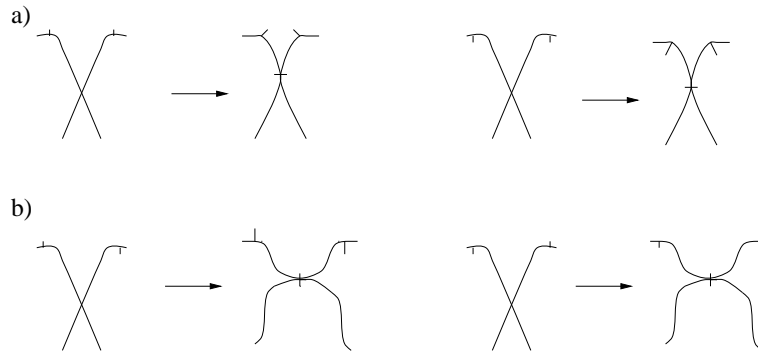


FIGURE 24.

the fibration by the \mathbb{Z}_2 -action, the projection of the deformed λ is not a cooriented

front anymore but a front equipped with a normal field of lines. (This corresponds to the factorization $S^1 \rightarrow \mathbb{R}P^1$.) Using Figure 25 one calculates the contributions of different cusps and double points to the total rotation number of the line field under traversing the boundary in the counter clockwise direction.

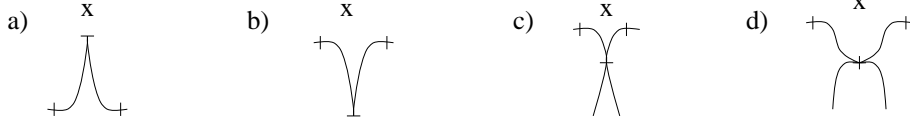


FIGURE 25.

These contributions are as follows:

$$\begin{cases} 1 & \text{for every cusp point pointing inside } X; \\ -1 & \text{for every cusp point pointing outside } X; \\ -1 & \text{for every double point of the type shown in Figure 8c;} \\ 0 & \text{for the other types of double points.} \end{cases} \quad (34)$$

To get the contributions to gleams, we divide these numbers by 2 (as we do in the construction of shadows, see 2.1).

If the region does not have cusps and double points in its boundary, then the obstruction to an extension of the section over ∂X to X is equal to $\chi(\text{Int } X)$.

This completes the proof of Theorem 5.2.3. \square

7.6. Proof of Theorem 5.3.9. A straightforward verification shows that

$$\text{ind } \tilde{L}_u^+ - \text{ind } \tilde{L}_u^- = \text{ind } L_u^+ - \text{ind } L_u^- - \epsilon(u), \quad (35)$$

$$\text{ind } \tilde{L}_u^+ + \text{ind } \tilde{L}_u^- = \text{ind } L + 1, \quad (36)$$

and

$$\text{ind } L_u^+ + \text{ind } L_u^- = \text{ind } L \quad (37)$$

for any double point u of L .

Let us prove (24). We write down the formal sums used to define $S(\lambda)$ and $l_q(\lambda)$ and start to reduce them in a parallel way as described below.

We say that a double point is essential if $[\tilde{\lambda}_u^+] \neq [\tilde{\lambda}_u^-]$.

For non-essential u we see that the term $([\tilde{\lambda}_u^+] - [\tilde{\lambda}_u^-])$ in $S(\lambda)$ is zero. Using (35) we get that the term $[\text{ind } L_v^+ - \text{ind } L_v^- - \epsilon(v)]_q$ in $l_q(\lambda)$ is also zero.

The index of a wave front coincides with the homology class of its lifting under the natural identification of $H_1(ST^*F)$ with \mathbb{Z} . This fact and (36) imply that if we have $[\tilde{\lambda}_u^+] = [\tilde{\lambda}_v^-]$ for two double points u and v , then $[\tilde{\lambda}_u^-] = [\tilde{\lambda}_v^+]$. Hence $([\tilde{\lambda}_u^+] - [\tilde{\lambda}_u^-]) = -([\tilde{\lambda}_v^+] - [\tilde{\lambda}_v^-])$, and these two terms cancel out. Identity (35) implies that the terms $[\text{ind } L_u^+ - \text{ind } L_u^- - \epsilon(u)]_q$ and $[\text{ind } L_v^+ - \text{ind } L_v^- - \epsilon(v)]_q$ also cancel out.

For similar reasons, if for a double point u the term $([\tilde{\lambda}_u^+] - [\tilde{\lambda}_u^-])$ is equal to $([\lambda] - f)$, so that we can simplify $S(\lambda)$ by crossing out the term and decreasing the coefficient C^+ by one. Then $[\text{ind } L_u^+ - \text{ind } L_u^- - \epsilon(u)]_q = [\text{ind } L - 1]_q$, and we can simplify $l_q(\lambda)$ by crossing out the term and decreasing the coefficient C^+ by one.

Similarly if the input of double point u into $S(\lambda)$ is $(e - [\lambda]f)$, then we reduce the two sums in the parallel way by crossing out the corresponding terms and decreasing by one the coefficients C^- .

We make the cancellations described above in both $S(\lambda)$ and $(l_q(\lambda) - [h]_qh)$ in a parallel way until we can not reduce $S(\lambda)$ any more. In this reduced form the terms of the form $k_l f^l$ with $k_l > 0$ correspond to the terms of type $[\tilde{\lambda}_u^+]$ for some double points u of L . (The case where a term of this type correspond to cusps is treated separately below.) Identities (35) and (36) imply that the contribution of the corresponding double points into $l_q(\lambda)$ is $k_l[2l - h - 1]_q$.

In the case where $k_l f^l$ term comes from the cusps and not from the double points of L , one can easily verify that the corresponding input of cusps into $(l_q(\lambda) - [h]_qh)$ can still be written as $k_l[2l - h - 1]_q$.

Thus $l_q(\lambda) = [h]_qh + \sum_{k_l > 0} k_l[2l - h - 1]_q$, and we have proved (24).

Let us prove (25). As above we reduce $S(\lambda)$ and $(l_q(\lambda) - [h]_qh)$ in a parallel way. Note that the coefficient at each $[m]_q$ was positive from the very beginning by the definition of $l_q(\lambda)$, and it stays positive under the cancellations described above. After this reduction each term $n_m[m]_q$ is a contribution of n_m double points. (The case where it is a contribution of cusps is treated separately as in the proof of (25).) Let u be one of these double points. Then from (35) and (36) we get the following system of two equations in variables $\text{ind } \tilde{L}_u^+$ and $\text{ind } \tilde{L}_u^-$:

$$\begin{cases} \text{ind } \tilde{L}_u^+ - \text{ind } \tilde{L}_u^- = m \\ \text{ind } \tilde{L}_u^+ + \text{ind } \tilde{L}_u^- = \text{ind } L + 1. \end{cases} \quad (38)$$

Solving the system we get that $[\tilde{\lambda}_u^+] = f^{\frac{m+h+1}{2}}$ and $[\tilde{\lambda}_u^-] = f^{\frac{h+1-m}{2}}$.

This proves identity (25) and Theorem 5.3.9. \square

7.7. Proof of Theorem 6.2.1. There are five elementary isotopies of a generic front L on an orbifold F . Four of them are: the birth of two cusps, passing through a non-dangerous self-tangency point, passing through a triple point, and passing of a branch through a cusp point. For all possible oriented versions of these moves a straightforward calculation shows that $S(\lambda) \in \frac{1}{2}\mathbb{Z}[H_1(N)]$ is preserved.

The fifth move is more complicated. It corresponds to a generic passing of a wave front lifted to \mathbb{R}^2 through the preimage of a cone point a . We can assume that this move is a symmetrization by G_a of the following move. The lifted front in the neighborhood of a is an arc C of a circle of large radius with center at the y axis, and during this move this arc slides through a along y (see Figure 26).

Clearly after this move points u and v on the arc C turn out to be in the same fiber if and only if they are symmetric with respect to the y axis, and the angle formed by v, a, u is less or equal to π and is equal to $\frac{2k\pi}{\mu_a}$ for some $k \in \{1, \dots, \mu_a\}$ (see Figure 26). We denote the set of such numbers k by $\bar{N}_1(a) = \{k \in \{1, \dots, \mu_a\} \mid \frac{2k\pi}{\mu_a} \in (0, \pi]\}$.

Two points u and v on the arc C are in the same fiber before the move if and only if they are symmetric with respect to the y axis, and the angle formed by u, a, v is less than π and is equal $\frac{2k\pi}{\mu_a}$ for some $k \in \{1, \dots, \mu_a\}$ (see Figure 26). We denote the set of such numbers k by $\bar{N}_2(a) = \{k \in \{1, \dots, \mu_a\} \mid \frac{2k\pi}{\mu_a} \in (0, \pi)\}$.

The projection of this move for the orientation of L' drawn in Figure 26 is shown in Figure 27.

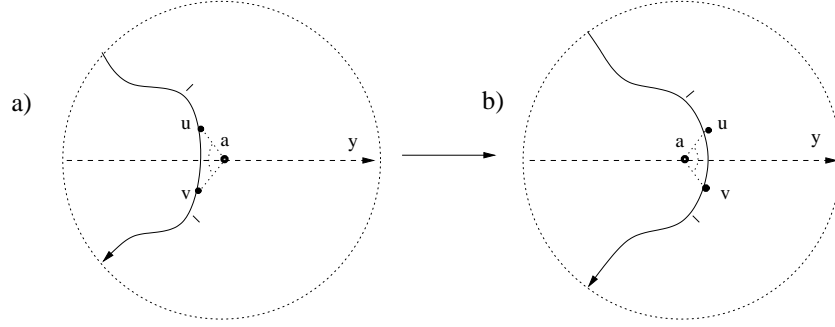


FIGURE 26.

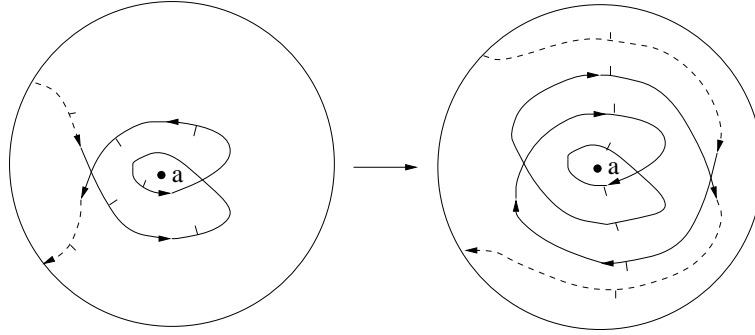


FIGURE 27.

Split the wave front in Figure 27 at the double point v (appearing after the move) that corresponds to some $k \in \bar{N}_1(a)$. Then $\tilde{\lambda}_v^-$ is a front with two positive cusps that rotates k times around a in the clockwise direction. Hence $[\tilde{\lambda}_v^-] = f f_a^{-k} = f_a^{\mu_a - k}$. We know that $[\tilde{\lambda}_v^+][\tilde{\lambda}_v^-] = [\lambda]f$ and that $f_a^{\mu_a} = f$. Thus $[\tilde{\lambda}_v^+] = [\tilde{\lambda}]f_a^k$.

In the same way we verify that if we split the front at the double point v (existing before the move) that corresponds to some $k \in \bar{N}_2(a)$, then $[\tilde{\lambda}_v^+] = f_a^k$ and $[\tilde{\lambda}_v^-] = [K]f_a^{|\mu_a| - k}$.

Now making sums over all corresponding numbers $k \in \{1, \dots, \mu_a\}$ we get that under this move $S(\lambda)$ changes by

$$\bar{R}_a^1 = \sum_{k \in \bar{N}_1(a)} ([\lambda]f_a^k - f_a^{\mu_a - k}) - \sum_{k \in \bar{N}_2(a)} (f_a^k - [\lambda]f_a^{\mu_a - k}). \quad (39)$$

A straightforward verification shows that $R_a^1 = \bar{R}_a^1$. (Note that the sets $N_1(a)$ and $N_2(a)$ are different from $\bar{N}_1(a)$ and $\bar{N}_2(a)$.)

Recall that $R_a^1 = 0 \in J$. Thus $S(\lambda)$ is invariant under the move.

For the other choice of the orientation of C the value of $S(\lambda)$ changes by $R_a^2 = 0 \in J$.

Hence $S(\lambda)$ is invariant under all elementary isotopies, and we have proved Theorem 6.2.1. \square

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